

CHAPTER - 3

PART 6 · ≈ 65 MIN

Forces in Rotating Machine Geometry — $r-\theta$

$n-t$ follows the path. $r-\theta$ follows the machine geometry. When a rotating arm sets how a part is allowed to move, the equations tell you what the actuator must supply.

6

Observe

›

Idealize

›

Model

›

Solve / Analyze

›

Validate

›

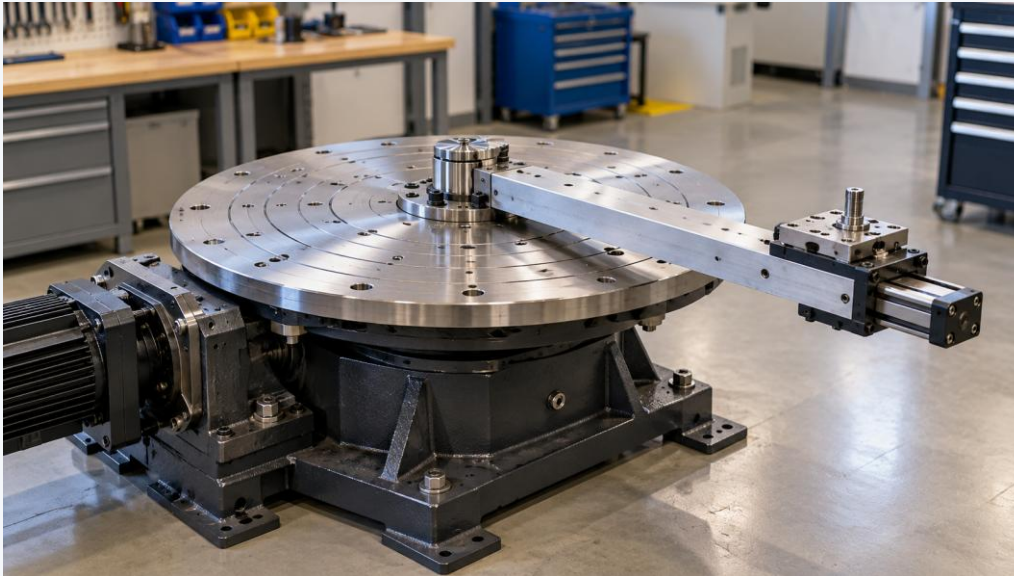
Report



One rotation, one radial motion, and a force nobody expected.

ROTATING FIXTURE WITH A RADIAL SLIDE

The arm turns about a fixed axis while the carriage slides along it. Two motions, one machine.



Why this Part exists

Automotive, mobility, manufacturing and automation are full of arms that rotate while something runs out along them. This is the coordinate system those machines are built in.



The engineering question

Not where the part goes — what the slide, the bearing and the actuator have to carry while it gets there.



The one term you can feel but cannot name

Walk outward along the radius of a spinning roundabout and you must push sideways, hard, in a direction that is not toward the centre. That push is this Part's main subject.

Observe



Idealize



Model



Solve / Analyze



Validate



Report



$n-t$ and $r-\theta$ are not harder and easier versions of each other. They differ in kind.

	What the axes follow	Where the origin is	Typical setting
$n-t$	the path the body traces	no origin; the axes ride on the body and turn with it	a known track, road, wire or trajectory
$r-\theta$	the geometry of the machine	a fixed, physically meaningful centre chosen by the engineer	a rotating arm, slide, fixture or manipulator



When you would choose $r-\theta$

When something real rotates about a definite centre, and something else moves along the rotating member. The machine gives you the coordinates — you do not have to invent them.



When you would not

When the path is given and there is no meaningful centre. A car on a road has a path but no pivot; use $n-t$ and let the axes ride.

Observe



Idealize

**Model**

Solve / Analyze



Validate



Report



The acceleration components in r- θ are not what they look like.

POLAR VELOCITY AND ACCELERATION

$$\vec{v} = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{u}_\theta$$

1

Radial acceleration is more than the second derivative of r

There is a second term, and it does not vanish just because the radial motion is steady.

2

Transverse acceleration is more than r times angular acceleration

There is a second term there too, and it survives even when the rotation rate is perfectly constant.



Just-in-time review, one slide

Both extra terms come from differentiating unit vectors that turn with the arm: each one changes at a rate θ times the other. They are physics, not typing errors.

Observe



Idealize



Model



Solve / Analyze



Validate

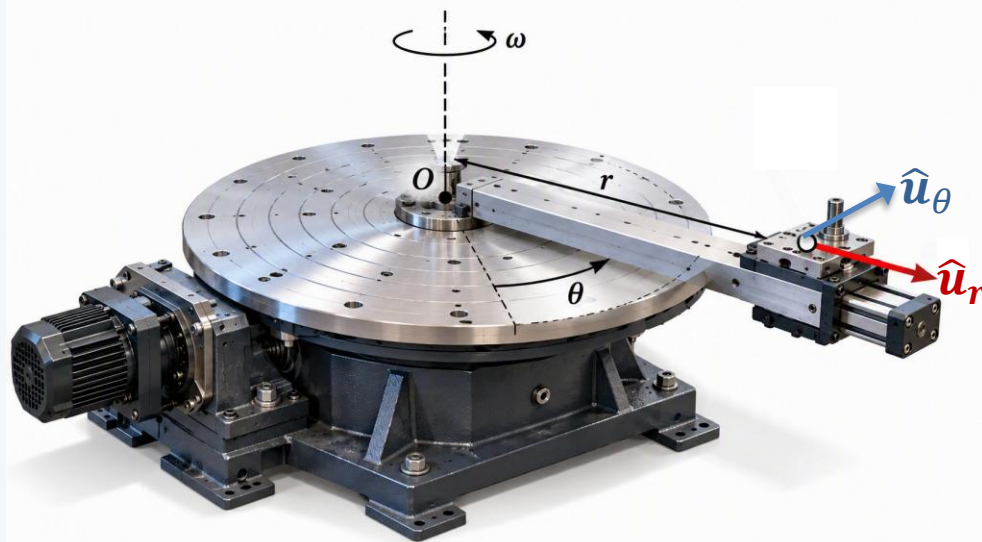


Report



Set the radial rates to zero and something familiar comes back.

CARRIAGE CLAMPED AT A FIXED RADIUS



SET $\dot{r} = 0$ AND $\ddot{r} = 0$

$$a_r = -r\dot{\theta}^2, \quad a_\theta = 0$$

This is the inward acceleration you already know, wearing new notation. The minus sign says the acceleration points toward O — opposite to increasing r .

The arm carries a radial force; nothing is needed along θ .
In the figure, ω is the same thing as $\dot{\theta}$.



Anchoring the new notation to something already understood

Every student can check this stage against Part 5. If it did not reduce to the familiar inward acceleration, the notation would be wrong.

WE17 · A 2.5 kg carriage clamped at $r = 0.40$ m on a fixture turning at 12 rad/s: $F_r = -m r \dot{\theta}^2 = -144$ N. The clamp is sized from that number.

Observe



Idealize



Model



Solve / Analyze



Validate



Report



Two more terms, each earned by one physical change.

STAGE 2 · THE ARM SPEEDS UP

$$a_{\theta} = r\ddot{\theta}$$

Angular acceleration of the mechanism. The drive has to supply a transverse force to produce it — which is what sizes the motor.

STAGE 3 · THE PART SLIDES OUTWARD

$$a_{\theta} = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

With $\theta = 0$, one transverse term survives: $2\dot{r}\dot{\theta}$. The part moves along r while the r direction itself rotates.



This is the roundabout push, in symbols

Walk outward on a spinning roundabout and you must be pushed sideways. Half of $2\dot{r}\dot{\theta}$ builds the extra tangential speed $r\dot{\theta}$ your new radius demands; the other half swings your outward velocity $\dot{r}$ round with the radial line.

$\theta = 0$ does not make the transverse acceleration zero. That is the entire content of the second term.

Observe



Idealize



Model



Solve / Analyze



Validate



Report



PREDICT

A carriage is driven outward along a rotating arm at a constant radial rate, while the arm turns at a constant rate.

Nothing is speeding up or slowing down. Is there a transverse force on the carriage?



Most people say no. Commit to an answer — the worked example after the equations settles it.

Observe



Idealize



Model



Solve / Analyze



Validate



Report



Four acceleration terms, four physical causes, two equations.

r - θ EQUATIONS OF MOTION

$$\sum F_r = m (\ddot{r} - r\dot{\theta}^2)$$

$$\sum F_\theta = m (r\ddot{\theta} + 2\dot{r}\dot{\theta})$$



What supplies each side

Along r : the radial drive, clamp or end-stop.
Along θ : the guide walls, backed by the arm and its motor. Every term is somebody's design load.

Term	What it is	Where the machine feels it
$\ddot{r}$	the part speeding up along the arm	the radial drive
$-r\dot{\theta}^2$	the inward requirement of circling O	radial drive, clamp or end-stop pulling inward
$r\ddot{\theta}$	the arm itself speeding up	guide walls, and the motor torque behind them
$2\dot{r}\dot{\theta}$	radial motion in a rotating direction	guide walls — even at constant θ

Observe



Idealize



Model



Solve / Analyze



Validate



Report



Constant rates everywhere — and still a 24 N sideways load.

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{u}_\theta$$

THE MODEL

Particle	the carriage and its payload, 6.0 kg
Description	r-θ about the fixture axis — the machine gives the coordinates
Motion	arm at a constant 8.0 rad/s; carriage driven outward at a constant 0.25 m/s
At the instant	r = 0.50 m
Assumptions	horizontal plane (weight balanced vertically); no friction along the slide
Unknowns	the radial force from the drive and the transverse force from the guide walls

SOLVE

$$a_r = 0 - 0.50(8.0)^2 = -32.0 \text{ m/s}^2$$

$$a_\theta = 0 + 2(0.25)(8.0) = 4.00 \text{ m/s}^2$$

$$F_r = -192 \text{ N}$$

$$F_\theta = 24.0 \text{ N}$$

Illustrative values. Negative F_r points toward O.



Read what the two numbers mean — and answer the Predict slide

The drive must pull the carriage inward with 192 N even though it is moving outward. The guide must push it sideways with 24 N: yes, there is a transverse force. Stop the radial motion and that 24 N disappears, though the arm turns just as fast.

Observe



Idealize



Model



Solve / Analyze



Validate

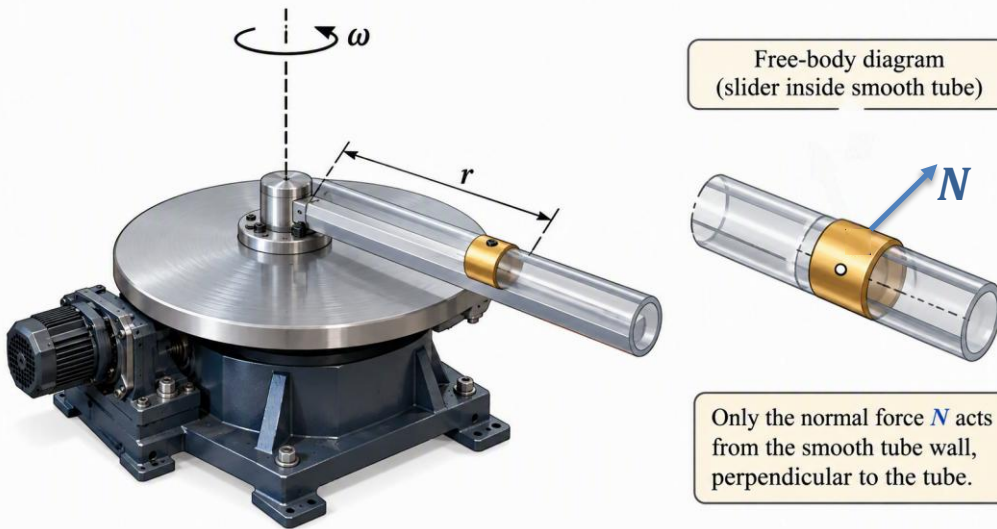


Report



No force along the tube. So is the radial acceleration zero?

SMOOTH ROTATING TUBE



SOLVE:

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{u}_\theta$$

Smooth, horizontal tube; the motor holds $\vartheta = \omega$ constant. No force acts along r :

$$\sum F_r = 0 = m(\ddot{r} - r\dot{\theta}^2)$$

$$\Rightarrow \ddot{r} = r\dot{\theta}^2$$

Zero radial force, yet $\ddot{r} = r\omega^2 > 0$. The farther out the part is, the faster it speeds up along the tube.



The dominant misconception, refuted

a_r is not $\ddot{r}$. Here $a_r = 0$ exactly, while $\ddot{r} = r\omega^2$ keeps growing.



And the transverse equation is busy too

With $\theta = 0$ the wall must push sideways with $N = 2m\dot{r}\omega$ — a load that grows as the part speeds up.

Observe



Idealize



Model



Solve / Analyze



Validate

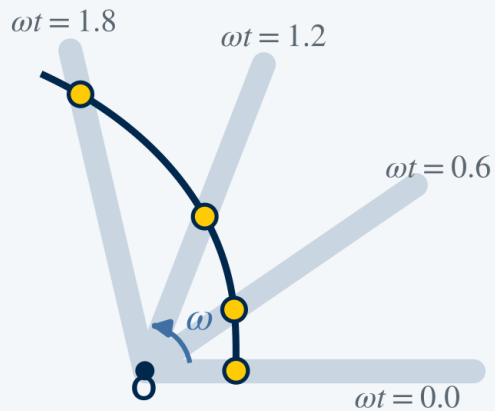


Report



The payoff of this Part is a picture you cannot get without computation.

THE FIXED-FRAME PATH



Seen from above

grey: tube at four instants
yellow: the part in each
navy: its fixed-frame path

Released at rest at r_0 ;
tube turns at constant ω

$$r = r_0 \cosh \omega t$$

ACTIVITY C6

1. Integrate both polar equations numerically (smooth tube, constant ω).
2. Convert $r(t)$, $\theta(t)$ to x , y and plot the path in the fixed frame.
3. Animate it with the tube drawn in, so the rotation is visible.
4. Check: with $\omega = 0$, a part released at rest must not move; with ω constant, $r(t)$ must match $r_0 \cosh \omega t$.



MEASURE · for under thirty dollars

Tape a tube to a turntable, film from directly above, and track $r(t)$ and $\theta(t)$. Compare the measured $r(t)$ with the prediction.

AI as coding companion

Fine for the integrator and the animation. The model and the reading of the result stay with you.

Observe



Idealize



Model



Solve / Analyze



Validate



Report



Design and prediction, in $r-\theta$.

DESIGN · INFERENCE

desired
motion



acceleration
components



required
forces

Specify the motion profile the machine must execute, and find out what the actuator, the slide and the bearing have to supply. This is how the machine gets sized.

PREDICTION · FORWARD

known
forces



acceleration
components



predicted
motion

Release a part in a rotating tube and find out where it goes and how fast. Same equations, read the other way.



And the same language discipline

The desired motion tells us what force is required. The acceleration never causes the force.

Whichever direction you are working in, write the frame and the origin on the page first. $r-\theta$ has an origin, and it matters.

Observe



Idealize



Model



Solve / Analyze



Validate



Report



Items 1–3 are one misconception in three disguises.

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{u}_\theta$$

1 “ a_r is just $\ddot{r}$.”

The dominant error in this Part. In the free-sliding tube, $a_r = 0$ while $\ddot{r} = r\dot{\theta}^2$.

Two cases:

- 1) no acceleration in radial direction, while $\ddot{r} \neq 0$.
- 2) $\ddot{r} = 0$ but radial acceleration can be $a_r = -r\dot{\theta}^2 \neq 0$

2 “Steady rotation means no transverse force.”

False whenever the radial rate is non-zero. That is the whole content of the transverse term, $a_\theta = 2\dot{r}\dot{\theta}$.

Steady Rotation means: $\ddot{\theta} = 0$

3 “The $2\dot{r}\dot{\theta}$ term is a mathematical correction.”

It is a design load with an actuator behind it. You can feel it on a roundabout, and you can measure it on a slide.

4 “Polar is just harder n–t.”

They differ in kind. One follows the path and has no origin; the other follows the machine and is anchored to a fixed centre that means something physically.

5 “The part flies outward because a force pushes it out.”

Part 5's conversation in a new geometry. In the fixed frame the tube wall pushes the part sideways, and nothing pushes it out.

$$\sum F_r = 0 = m(\ddot{r} - r\dot{\theta}^2)$$

This is tied to [1] above.

Observe



Idealize



Model



Solve / Analyze



Validate



Report



Most of them cost one line.

1 Kill the rotation

Set $\dot{\theta} = 0$ and $\ddot{\theta} = 0$. The equations must reduce to straight-line motion along a fixed radial line.

2 Kill the radial motion

Set $\dot{r} = 0$ and $\ddot{r} = 0$. The equations must reduce to the Part 5 circular-motion result, inward force and all.

3 Check the signs physically

Does the transverse force point the way the mechanism is actually pushing? Does the radial force point inward when it should?

4 Check the units, term by term

$r \theta^2$ is $\text{m} \times (\text{rad/s})^2 = \text{m/s}^2$ because radians are dimensionless — which is also the reason every derivative in this Part must be taken with θ in radians.



Degrees will silently ruin it

If a problem states an angular rate in revolutions per minute or degrees per second, convert before differentiating. This is the most common numerical error in polar kinetics and it leaves no trace in the algebra.

Observe



Idealize



Model



Solve / Analyze



Validate



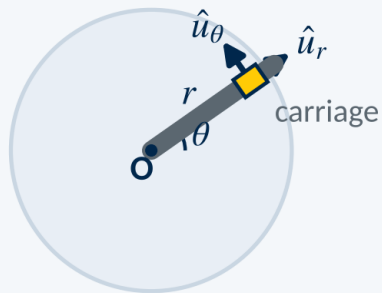
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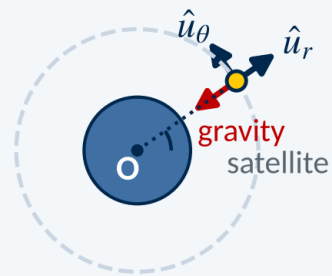
Part 6 — and one slide about scale before we close.

A MACHINE AND A SATELLITE, SIDE BY SIDE

Rotating fixture (top view)



Satellite about the Earth



The same two equations describe a 6 kg carriage on a workshop fixture and a satellite in orbit.

Same dynamics. Different application.



You can now EXPLAIN

- why r - θ follows the machine, not the path
- the physical origin of all four acceleration terms
- why radial acceleration is not the second derivative of r



You can now DO

- choose between x - y , n - t and r - θ and defend it
- size an actuator or slide load from a motion profile
- integrate the polar equations and read the fixed-frame path

Observe



Idealize



Model



Solve / Analyze



Validate



Report



Six Parts of models, and none of them questioned yet.

Part	What it gave you	What it assumed without saying much
3 · Cartesian	cause and effect, both directions	a known drive force, a level surface, no aerodynamics
4 · Connections	compatibility and closure	a massless cable, a frictionless pulley, a rigid coupling
5 · Curved path	the two components and contact loss	a single coefficient of friction, a rigid road, a point mass
6 · Machine geometry	the four terms and their loads	a smooth slide, a rigid arm, no clearance

Every one of those assumptions was written down and then stopped being thought about. Part 7 finds out which of them mattered.

Next — Part 7: my equations gave me an answer. Should I trust it?

Observe



Idealize



Model



Solve / Analyze



Validate

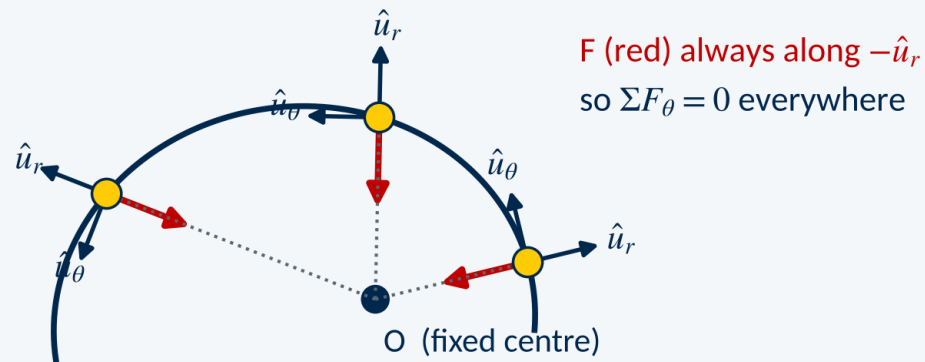


Report



Gravity is a central force — and that geometry alone determines a great deal.

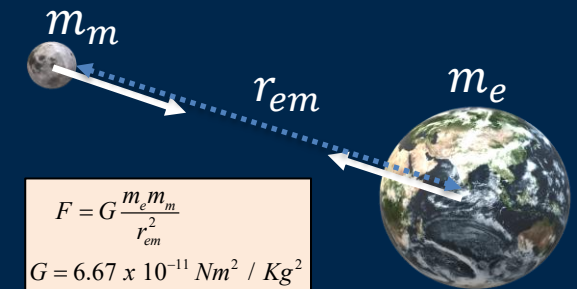
A CENTRAL FORCE



NEWTON'S LAW OF GRAVITATION

$$F = G \frac{m_1 m_2}{r^2}$$

Always directed along the line joining the two bodies. It has a radial component and nothing else — which is the only fact this appendix needs.



$$F = G \frac{m_e m_m}{r_{em}^2}$$

$G = 6.67 \times 10^{-11} \text{ Nm}^2 / \text{Kg}^2$
 $m_e = 6.0 \times 10^{24} \text{ Kg}$
 $m_m = 7.35 \times 10^{22} \text{ Kg}$
 $r_{em} = 384,400,000 \text{ m}$
 $\Rightarrow F = 1.99 \times 10^{20} \text{ N}$



Why this is an appendix and not the live core

The live Part is built on rotating machinery, because that is where these students will work. This material is enrichment: a self-study session, an extra-credit thread, or one lecture in a semester that runs ahead.

Observe



Idealize



Model



Solve / Analyze



Validate



Report



No transverse force means a quantity that does not change.

SET THE TRANSVERSE SUM TO ZERO

$$\sum \vec{F}_\theta = 0 \Rightarrow r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

Multiply by r and the left side is a perfect derivative:

$$\frac{d}{dt} (r^2 \dot{\theta}) = 0 \Rightarrow r^2 \dot{\theta} = \text{constant}$$



A conserved quantity, from an equation of motion

Nothing was assumed about the path. The geometry of the force did all the work — and this is the first conserved quantity in the course.



LOOKING AHEAD — named, not taught

That constant is the angular momentum per unit mass. Chapter 5 defines it properly and shows what it is good for. Here it is only a preview: nothing in this appendix is solved with it.

Optional appendix. Not delivered in the live lecture, and not examinable as part of the Chapter 3 core.

Observe



Idealize



Model

**Solve / Analyze**

Validate



Report



Set the radius constant and the radial equation does the rest.

SOLVE

A circular orbit has r constant, so $\dot{r} = 0$ and $\ddot{r} = 0$:

$$G \frac{Mm}{r^2} = mr\dot{\theta}^2 \quad \rightarrow \quad v_{sp} = \sqrt{\frac{GM}{r}}$$

The orbital speed depends only on the radius, not on the satellite's mass.



Nothing new was needed

The Part 6 radial equation with $\dot{r} = \ddot{r} = 0$, and gravity as the force.



What is deliberately excluded

Ellipses, Kepler's laws, escape speed, transfers: all need Chapter 4–5 methods.

Every worked case in this appendix is circular, so the conserved quantity is never actually needed.

Observe



Idealize



Model



Solve / Analyze



Validate



Report



Three lines of algebra, and an answer you can look up.

SOLVE

For the satellite to stay above one point, its period must equal one rotation of the Earth:

$$r = \sqrt{\frac{GMT^2}{4\pi^2}}$$

With $T = 23 \text{ h } 56 \text{ min}$ this gives $r \approx 4.22 \times 10^7 \text{ m}$ from the centre of the Earth — about 35 800 km of altitude.



Validate it

Compare with the published value: three figures from three lines of algebra.



And note the period used

Sidereal, not solar: 24 h moves r by $\approx 0.2 \%$ ($\approx 80 \text{ km}$) — a parameter error.



End of the optional appendix

Four slides, about 20 to 25 minutes if delivered. Not counted in the chapter's core time budget, and not a prerequisite for anything in Part 7.

Observe



Idealize



Model



Solve / Analyze



Validate



Report