

**AI-AUGMENTED ENGINEERING DYNAMICS:
A CONVERSATIONAL APPROACH**

Chapter 5 — Momentum Methods for Particles

Taner Onsay, Ph.D.

Author

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Observe



Idealize



Model



Solve / Analyze



Validate



Report

The engineering workflow you will use throughout this book

CHAPTER - 5

PART 5 · ≈ 60 MIN

What Is Angular Momentum?

Two balls with the same momentum can relate to a point in completely different ways. This Part measures that relationship, angular momentum about a point, and shows what makes it change.

5

Observe

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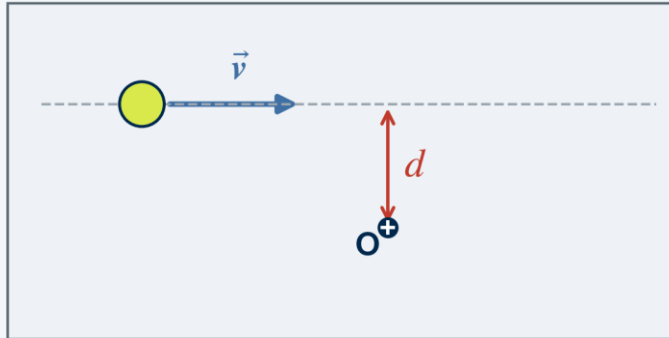
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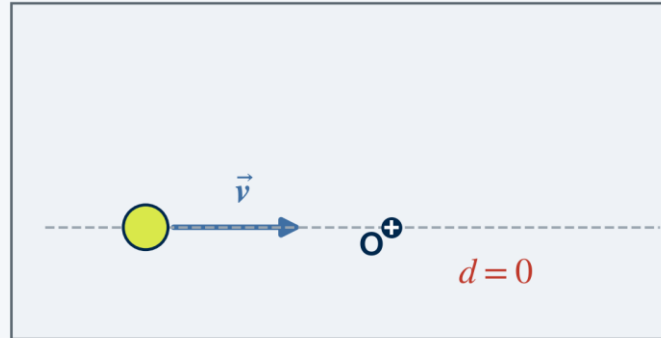
Same Momentum, Different Relationship to O

SCHEMATIC · NOT A MEASUREMENT

Case A: path passes beside O



Case B: path aimed straight at O



LIVE DEMO

- Mark O on the desk.
- Roll the ball past O, then straight at O, at about the same speed.
- Ask: did the ball go around O?

From Part 4: if an impact force's line of action misses a reference point, can it change something besides linear momentum? **Today: yes — angular momentum about that point.**

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Three Predictions About the Two Rolls

PREDICTION 1

Can the two balls have the same linear momentum?

- A** yes
- B** no
- C** only if they hit O

Yes. Same mass, same velocity: $L^{\rightarrow} = m\mathbf{v}^{\rightarrow}$ is identical. Linear momentum ignores where the line of motion is.

PREDICTION 2

Do they have the same angular momentum about O?

- A** yes
- B** no
- C** only if the ball curves

No. The ball aimed at O has none about O. The ball passing beside O has some.

PREDICTION 3

What decides how much angular momentum about O?

- A** the speed alone
- B** the distance from O to the line of motion
- C** the mass alone

The perpendicular distance d from O to the line of motion, together with m and the speed.

Observe

Idealize

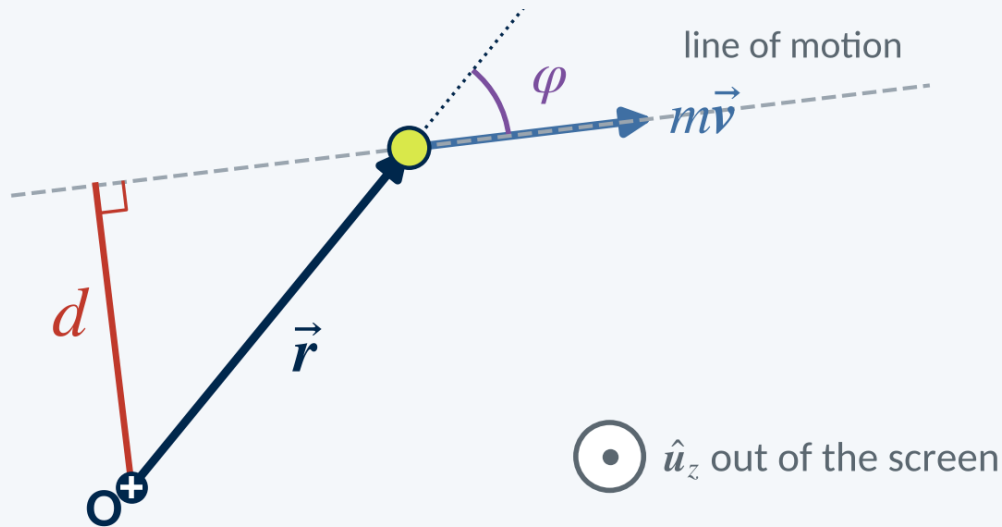
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Angular Momentum: Position Cross Momentum



ANGULAR MOMENTUM ABOUT A FIXED POINT O

$$\vec{H}_O = \vec{r} \times m\vec{v} = \vec{r} \times \vec{L}$$

planar motion: $\vec{H}_O = H_{Oz} \hat{u}_z$

$$H_{Oz} = (\vec{r} \times m\vec{v})_z$$

Units: $\text{kg} \cdot \text{m}^2/\text{s}$.

- $\vec{r}$ is drawn **from O** to the particle; choose and label O first.
- O is fixed in an inertial frame.
- A particle has no size, so no spin of its own: $\vec{H}_O$ comes entirely from its motion relative to O.

$\vec{L}$ tells how much the particle is moving; $\vec{H}_O$ tells how much that motion swings around O.

Observe

Idealize

Model

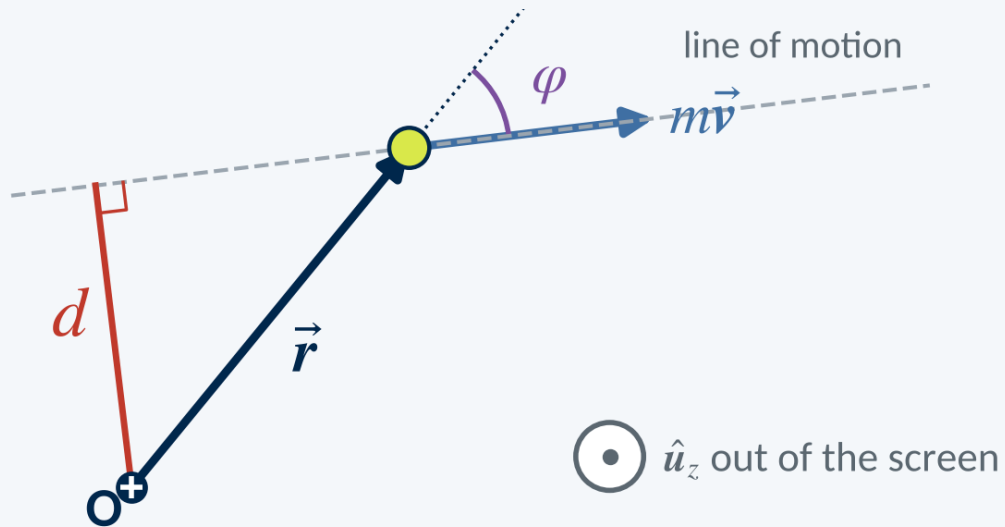
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Perpendicular Distance and Angle



Not restricted to circles: any straight path that misses O has nonzero angular momentum about O .

MAGNITUDE

$$|\vec{H}_O| = mrv_{sp} \sin \varphi = mv_{sp}d$$

- $r = |\vec{r}|$, distance from O ; v_{sp} = speed (≥ 0).
- φ : angle between $\vec{r}$ and $\vec{v}$.
- $d = r \sin \varphi$: perpendicular distance from O to the line of motion.

DIRECTION AND SIGN

- Right-hand rule: fingers along $\vec{r}$, curl toward $\vec{v}$; thumb gives $\vec{H}_O$.
- Counterclockwise about O : $H_{Oz} > 0$. Clockwise: $H_{Oz} < 0$.
- Line of motion through O : $d = 0$, so $\vec{H}_O = \mathbf{0}$.



Same Particle, Three Choices of O

YOUR PREDICTION

A particle moves in $+x$ along the line $y = 0.30$ m. What is the sign of H_{Oz} about each point?

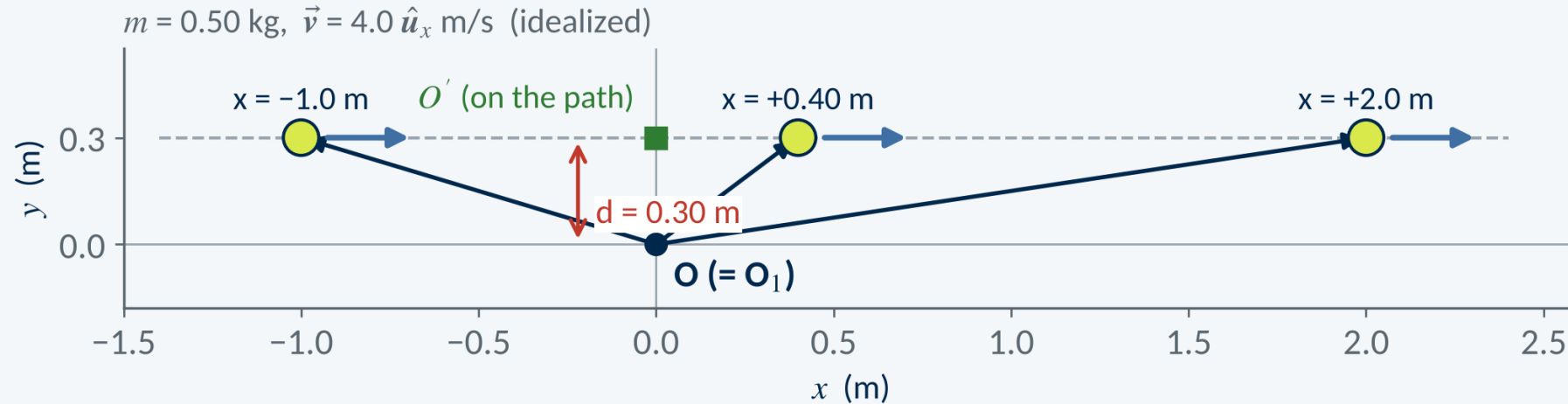
- 1 O_1 at (0, 0): below the path
- 2 O_2 at (0, 0.30 m): on the path
- 3 O_3 at (0, 0.60 m): above the path

ANSWER

- **O_1 : negative.** The particle passes above O_1 moving right: clockwise.
- **O_2 : zero.** O_2 lies on the line of motion, so $d = 0$.
- **O_3 : positive.** Passing below O_3 moving right: counterclockwise.
- Same $\vec{L}$, three different $\vec{H}_O$. The reference point is part of the answer.



A Straight-Line Particle, Two Reference Points



Position x (m)	$\vec{r}$ (m)	d (m)	H_{O_z} (kg·m ² /s)
-1.0	(-1.0, 0.30)	0.30	-0.60
+0.40	(0.40, 0.30)	0.30	-0.60
+2.0	(2.0, 0.30)	0.30	-0.60

$$|\vec{H}_O| = mv_{sp}d = (0.50)(4.0)(0.30) = 0.60 \text{ kg} \cdot \text{m}^2/\text{s}$$

- Clockwise about O, so $H_{O_z} = -0.60 \text{ kg} \cdot \text{m}^2/\text{s}$ at every x : $\vec{r}$ and φ change, d does not.
- About O' on the path (O_2 on the previous slide): $d = 0$, so $H_{O'_z} = 0$.



Computing the Cross Product in x and y

PLANAR MOTION, O AT THE ORIGIN

$$\vec{r} = x\hat{u}_x + y\hat{u}_y, \quad \vec{v} = v_x\hat{u}_x + v_y\hat{u}_y$$

$$H_{Oz} = m(xv_y - yv_x)$$

Signed components go in with their signs; the result is a signed H_{Oz} .

CHECK WITH EXAMPLE A ($x = 0.40$ m)

$$H_{Oz} = 0.50[(0.40)(0) - (0.30)(4.0)]$$

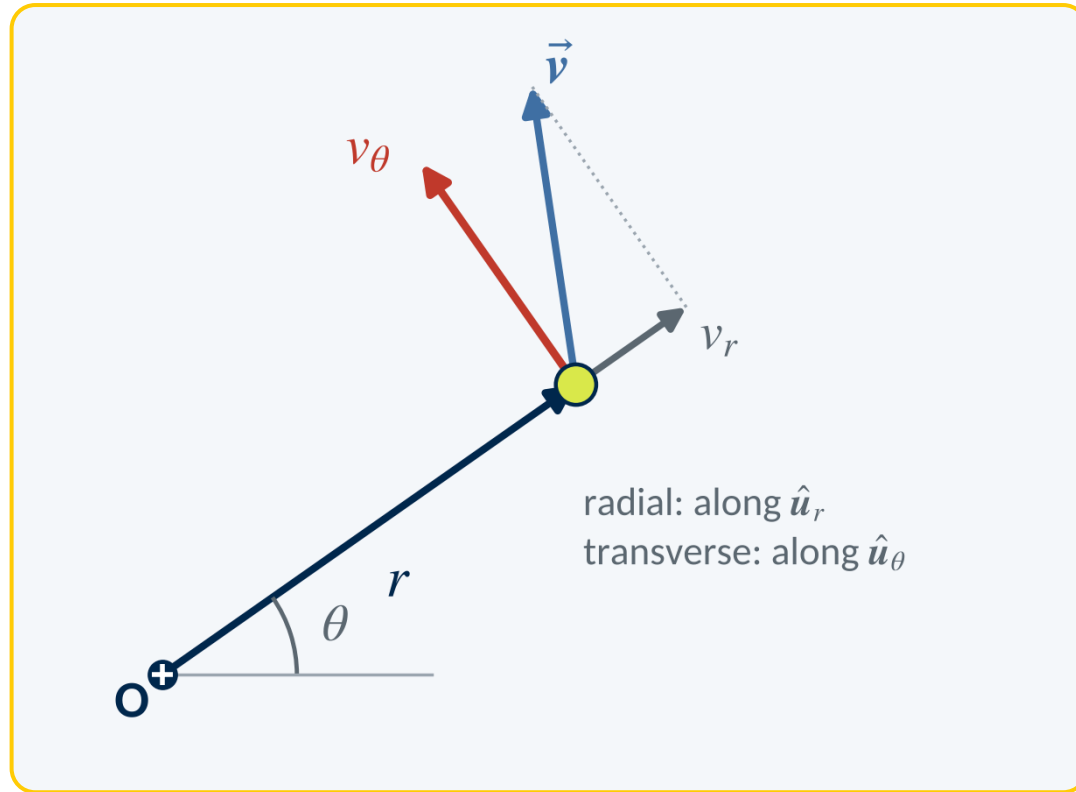
$$= -0.60 \text{ kg} \cdot \text{m}^2/\text{s} \checkmark$$

Matches the moment-arm result, sign included.

- xv_y : for $x > 0$, motion in $+y$ turns the particle counterclockwise about O (positive).
- $-yv_x$: for $y > 0$, motion in $+x$ turns it clockwise (negative).
- Use this form when coordinates are given; use $mv_{sp}d$ when the geometry is simple.



Only Transverse Velocity Counts



Transverse (θ), not tangential: tangential means along the velocity.

POLAR FORM (O AT THE POLE, FIXED)

$$\vec{v} = v_r \hat{u}_r + v_\theta \hat{u}_\theta, \quad v_r = \dot{r}, \quad v_\theta = r\dot{\theta}$$

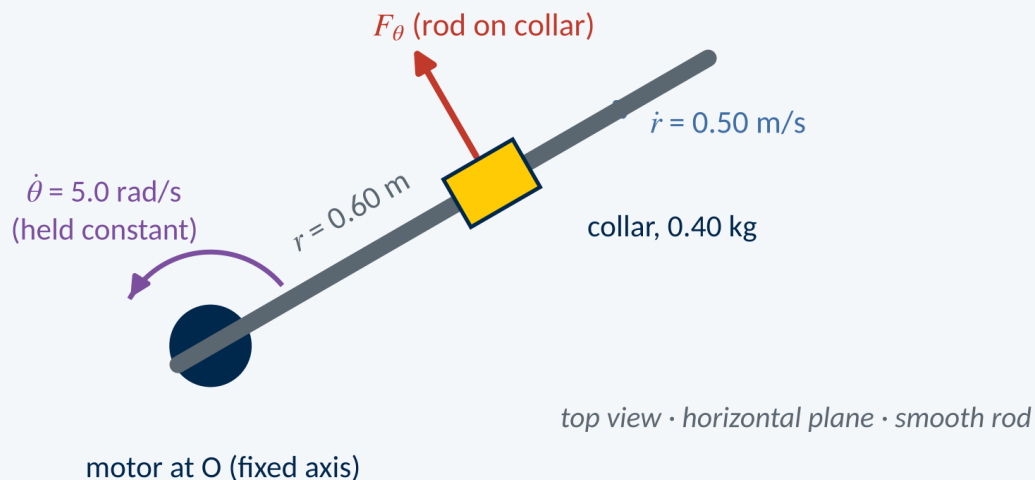
$$H_{Oz} = mrv_\theta = mr^2\dot{\theta}$$

$\vec{r} \times v_r \hat{u}_r = \mathbf{0}$: the radial part is parallel to $\vec{r}$.

- Radial velocity contributes **nothing directly** to H_{Oz} .
- Indirectly it matters: moving outward at the same θ raises $mr^2\dot{\theta}$, and that increase must be supplied by a transverse force (Example B).
- $mr^2\dot{\theta}$ is for a particle about O; it is not a rigid-body formula.



A Collar on a Motor-Driven Rotating Rod



- Idealized: $m = 0.40 \text{ kg}$, $r = 0.60 \text{ m}$, $\dot{r} = +0.50 \text{ m/s}$.
- Motor holds $\theta = +5.0 \text{ rad/s}$, so $\ddot{\theta} = 0$. Horizontal plane; smooth rod; collar as a particle.

$$v_{\theta} = r\dot{\theta} = (0.60)(5.0) = 3.0 \text{ m/s}, \quad v_r = 0.50 \text{ m/s}$$

$$H_{Oz} = mrv_{\theta} = (0.40)(0.60)(3.0) = 0.72 \text{ kg} \cdot \text{m}^2/\text{s}$$

$$\frac{dH_{Oz}}{dt} = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$$

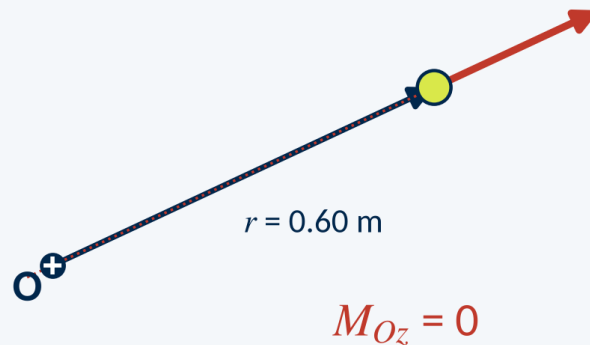
$$= (0.40)(2)(0.60)(0.50)(5.0) = 1.20 \text{ N} \cdot \text{m}$$

Motor-driven, not free: the motor keeps θ fixed. A freely spinning rod would slow down as the collar moves out; that is a different model.

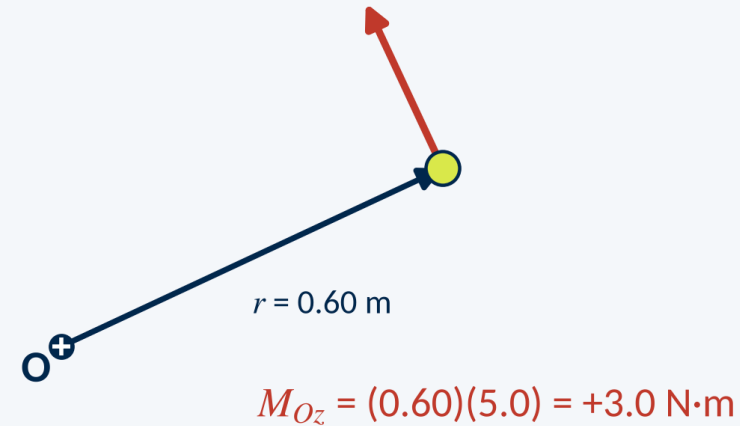


Direction and Line of Action Matter

radial 5.0 N: line of action through O



transverse 5.0 N: arm = r



$$\vec{M}_O = \vec{r} \times \vec{F}, \quad M_{Oz} = (\vec{r} \times \vec{F})_z$$

polar: $M_{Oz} = rF_\theta$

Units: N·m (same dimensions as the joule, but never written in joules).

- Same 5.0 N, same point: moment +3.0 N·m or 0.
- A nonzero force can have zero moment about a point: here, when its line of action passes through O.

Observe

Idealize

Model

Solve / Analyze

Validate

Report



Why the Net Moment Changes Angular Momentum

DIFFERENTIATE, CONSTANT m , FIXED O

$$\frac{d\vec{H}_O}{dt} = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times m \frac{d\vec{v}}{dt}$$

first term: $\vec{v} \times m\vec{v} = \vec{0}$ (parallel vectors)

second term: $\vec{r} \times m\vec{a} = \vec{r} \times \Sigma\vec{F} = \Sigma\vec{M}_O$

$$\Sigma\vec{M}_O = \frac{d\vec{H}_O}{dt}, \quad \Sigma M_{Oz} = \frac{dH_{Oz}}{dt}$$

$d\vec{r}/dt = \vec{v}$ only because O does not move.

WHAT IT SAYS

- The net moment about O sets how fast $\vec{H}_O$ changes, just as $\Sigma\vec{F} = d\vec{L}/dt$.
- Conditions: particle of constant mass; O fixed in an inertial frame.
- This is an instantaneous rate equation. What it implies over an interval is Part 6.



Radial Forces, Transverse Forces, Straight Lines



Radial force only

Line of action through O: $M_{Oz} = 0$, so $dH_{Oz}/dt = 0$ at that instant.



Transverse force

Has moment rF_{θ} : it changes H_{Oz} . In Example B, the rod's sideways push.



Radial motion with angular momentum

The collar moves outward and still has $H_{Oz} = 0.72 \text{ kg}\cdot\text{m}^2/\text{s}$ from its transverse velocity.



Straight line, no force

Example A: no in-plane force, no moment, H_{Oz} unchanged at $-0.60 \text{ kg}\cdot\text{m}^2/\text{s}$.

The rate of change of H_{Oz} is set by the net moment about the chosen fixed point, nothing else.

Observe ›

Idealize ›

Model ›

Solve / Analyze ›

Validate ›

Report



Does the Rod's Push Explain the Rate?

ROUTE 1 · FROM ANGULAR MOMENTUM

$$\frac{dH_{Oz}}{dt} = \frac{d}{dt}(mr^2\dot{\theta}) = m(2r\dot{r}\dot{\theta})$$

$$= (0.40)(2)(0.60)(0.50)(5.0) = 1.20 \text{ N} \cdot \text{m}$$

Uses only the kinematics of $H_{Oz} = mr^2\dot{\theta}$.

ROUTE 2 · FROM NEWTON, TRANSVERSE (θ)

$$F_{\theta} = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = (0.40)(2)(0.50)(5.0) = 2.0 \text{ N}$$

$$M_{Oz} = rF_{\theta} = (0.60)(2.0) = 1.20 \text{ N} \cdot \text{m}$$

Uses the polar acceleration and the force the rod must supply.

Both routes: 1.20 N·m. The $2\dot{r}\dot{\theta}$ term is why a rod turning at constant $\dot{\theta}$ must still push sideways.

Power check: $P = F_{\theta}v_{\theta} = (2.0)(3.0) = 6.0 \text{ W} = M_{Oz}\dot{\theta} = (1.20)(5.0)$. Ideal instantaneous mechanical power delivered to the collar; not a motor's electrical input.

Observe ›

Idealize ›

Model ›

Solve / Analyze ›

Validate ›

Report



Three Plausible but Wrong Explanations



AI CLAIM 1

“A particle moving in a straight line has zero angular momentum.”

WHAT IS WRONG

No reference point named. About a point off the line, $H_{Oz} = \pm m v_{sp} d \neq 0$ (Example A: $-0.60 \text{ kg}\cdot\text{m}^2/\text{s}$). Zero only about points on the line.



AI CLAIM 2

“Radial velocity contributes directly to angular momentum about O.”

WHAT IS WRONG

$\mathbf{r} \times v_r \hat{\mathbf{u}}_r = \mathbf{0}$: radial velocity is parallel to $\mathbf{r}$. Only v_θ enters: $H_{Oz} = m r v_\theta$.



AI CLAIM 3

“Because the rod is frictionless, it cannot exert a transverse force on the collar.”

WHAT IS WRONG

Frictionless removes force along the rod, not perpendicular to it. The rod's normal force is transverse: 2.0 N in Example B.

Observe ›

Idealize ›

Model ›

Solve / Analyze ›

Validate ›

Report



Momentum About a Point

	Linear	Angular, about fixed O
Quantity	$L^{\rightarrow} = m\boldsymbol{v}^{\rightarrow}$	$H^{\rightarrow}_O = \boldsymbol{r}^{\rightarrow} \times m\boldsymbol{v}^{\rightarrow}$
Rate equation	$\Sigma \boldsymbol{F}^{\rightarrow} = d\boldsymbol{L}^{\rightarrow}/dt$	$\Sigma \boldsymbol{M}^{\rightarrow}_O = d\boldsymbol{H}^{\rightarrow}_O/dt$
Depends on	mass and velocity	also position relative to O (moment arm d)
Units	kg·m/s	kg·m ² /s

Workflow: choose a fixed O → draw $\boldsymbol{r}^{\rightarrow}$ from O → compute $H^{\rightarrow}_O$ (moment arm, components, or polar) → find $\Sigma \boldsymbol{M}^{\rightarrow}_O$ → check two routes.

Next — Part 6: If the net moment about O is zero over an interval of time, what happens to angular momentum about O?

CHAPTER - 5 · PART 5

APPENDIX

Appendix — Supplemental Material

A1 cross-product geometry and 3-D components · A2 derivation for fixed O ·
A3 polar derivative · A4 motor-driven collar vs free rod · A5 changing the
reference point.

Observe



Idealize



Model



Solve / Analyze



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Right-Hand Rule and Three-Dimensional Components

COMPONENTS (O AT THE ORIGIN)

$$H_{Ox} = m(yv_z - zv_y)$$

$$H_{Oy} = m(zv_x - xv_z)$$

$$H_{Oz} = m(xv_y - yv_x)$$

Planar motion ($z = 0, v_z = 0$) leaves only H_{Oz} , the component used in the core.

PRACTICE (IDEALIZED)

- $m = 0.20 \text{ kg}$, $\mathbf{r}^{\rightarrow} = (1.0, 2.0, 0.5) \text{ m}$, $\mathbf{v}^{\rightarrow} = (3.0, -1.0, 2.0) \text{ m/s}$.
- $\mathbf{r}^{\rightarrow} \times \mathbf{v}^{\rightarrow} = (4.5, -0.5, -7.0) \text{ m}^2/\text{s}$.
- $\mathbf{H}_O = (0.90, -0.10, -1.40) \text{ kg}\cdot\text{m}^2/\text{s}$.

- Order matters: $\mathbf{r}^{\rightarrow} \times \mathbf{v}^{\rightarrow} = -\mathbf{v}^{\rightarrow} \times \mathbf{r}^{\rightarrow}$. Always position first, from O.
- Unit vectors: $\hat{\mathbf{u}}_x \times \hat{\mathbf{u}}_y = \hat{\mathbf{u}}_z$, $\hat{\mathbf{u}}_y \times \hat{\mathbf{u}}_z = \hat{\mathbf{u}}_x$, $\hat{\mathbf{u}}_z \times \hat{\mathbf{u}}_x = \hat{\mathbf{u}}_y$.

Observe ›

Idealize ›

Model ›

Solve / Analyze ›

Validate ›

Report



Why O Must Be Fixed

PARTICLE, CONSTANT MASS, INERTIAL FRAME

$$\frac{d\vec{H}_O}{dt} = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times m\vec{a}$$

$$\text{O fixed: } \frac{d\vec{r}}{dt} = \vec{v} \rightarrow \vec{v} \times m\vec{v} = \vec{0}$$

O moving at $\vec{v}_O$: extra term $-\vec{v}_O \times m\vec{v}$

With a moving point the simple form $\Sigma \vec{M}_O = d\vec{H}_O/dt$ fails unless that term vanishes.

ASSUMPTIONS USED

- Single particle, constant m .
- Newton's second law: inertial frame.
- O fixed in that frame.
- $\Sigma \vec{F}$ includes every force on the particle.

Moving reference points (for example a system's centre of mass) are treated later, with systems and rigid bodies. In this Part, always choose O fixed.

Observe ›

Idealize ›

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Angular Momentum Rate in r and θ

FROM THE POLAR FORM OF ANGULAR MOMENTUM

$$\frac{dH_{Oz}}{dt} = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$$

$$= r \cdot m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = r m a_{\theta}$$

$$= r F_{\theta} = \Sigma M_{Oz}$$

The angular-momentum rate equation and Newton's transverse equation are the same statement, multiplied by r .

READING THE TERMS

- $r^2\ddot{\theta}$: speeding up the rotation.
- $2r\dot{r}\dot{\theta}$: carrying the same rotation rate to a larger radius.
- Example B: only the second term, 1.20 N·m.

This is why slide 16's two routes must agree: they are one equation reached from two directions.

Observe ›

Idealize ›

Model ›

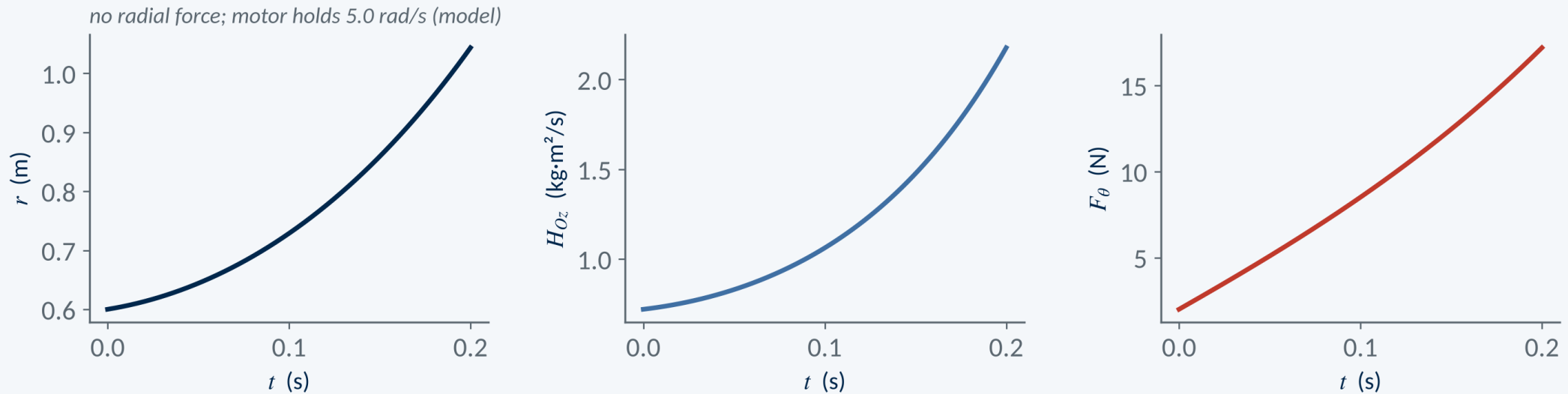
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What Happens Next to the Collar?



- Smooth rod, so $F_r = 0$: $\ddot{r} = r\dot{\theta}^2 = 15$ m/s² at the given instant; the collar is flung outward.
- Motor still holds 5.0 rad/s: by 0.20 s, $r \approx 1.04$ m, $H_{O_z} \approx 2.18$ kg·m²/s, $F_\theta \approx 17$ N (model).

- **Free rod (no motor):** the rod slows as the collar moves out.
- Predicting by how much needs the rod's own inertia: rigid-body material, later in the course.

Observe ›

Idealize ›

Model ›

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Same Momentum, Different Point

TRANSFORMATION

$$\vec{H}_{O'} = \vec{H}_O - \vec{r}_{O'} \times \vec{L}$$

$\vec{r}_{O'/O}$: position of the new point O' relative to O . Only the component of $\vec{r}_{O'/O}$ perpendicular to $\vec{L}$ matters.

CHECK ON EXAMPLE A

- $O' = (0, 0.30)$: $-0.60 - (-0.60) = 0 \checkmark$
- $O_3 = (0, 0.60)$: $-0.60 - (-1.20) = +0.60 \checkmark$
- O moved along $\vec{L}$ (e.g. to $(5, 0)$): no change.

CHALLENGE QUESTIONS

- Is there a point about which a particle moving in a straight line has angular momentum $+1.20 \text{ kg}\cdot\text{m}^2/\text{s}$? Where?
- Two particles with equal and opposite linear momenta: does their total angular momentum depend on O ?
- Why does choosing O on the line of a force make that force disappear from the moment equation?